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A note on mirror symmetry for manifolds with spin(7) holonomy

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Abstract
Starting from the extended worldsheet superconformal algebras associated with G_2 manifolds, we extend the algebra to Joyce's orbifolds with $\mathfrak{s}pin(7)$ holonomy. We show how the mirror symmetry in manifolds with $\mathfrak{s}pin(7)$ holonomy arises as the automorphism in the extended sperconformal algebra. In one class of Joyce's orbifolds, the automorphism of the superconformal algebra can be realized as 14 kinds of T-dualities along the supersymmetric T^4 toroidal fibrations in the manifolds with $\mathfrak{s}pin(7)$ holonomy. In this class of examples, Joyce's orbifolds are pairwise identified under the mirror symmetry. We then discuss some interesting features of the mirror symmetry on the manifolds with exceptional holonomy and how it is different from the Calabi–Yau mirror symmetry.

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1. Introduction

Mirror symmetry is a beautiful subject both in physics and mathematics. It was first conjectured in [6] that there exists a symmetry which exchanges the complex moduli on one manifold with the Kähler moduli on the dual manifold when we consider the string worldsheet propagation on Calabi–Yau (CY) manifolds. The symmetry arises in the sense that the resulting physical spectra of the mirror pair are isomorphic. This requires the Hodge numbers of the CY mirror pair satisfy the condition $b_{p,q}(M) = b_{d-p,q}(\tilde{M})$. It was also shown that, assuming the mirror symmetry, one could determine non-perturbative worldsheet instantons effect on CY M by computing classical periods in its mirror $\tilde{M}$ [20]. In other words, one can utilize mirror

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symmetry to compute worldsheet instanton correction to the worldsheet three-point function, namely, Yukawa couplings.

Another more intuitive perspective of mirror symmetry was proposed by Strominger, Yau and Zaslow (SYZ) [7]. SYZ argued that the mirror transformation in CY is in fact equivalent to the T-duality along the supersymmetric torus fibration in the CY manifolds, by considering the mirror BPS soliton spectra in two theories (IIA/IIB). The moduli space of the supersymmetric Langrangian branes is precisely the same as the mirror manifold. In this paper we will be generalizing this notion to other manifolds with exceptional holonomy. Since this paper is not about reviewing mirror symmetry and also it is such a huge subject, the readers interested in various aspects of mirror symmetry are referred to [18, 19].

Some concrete mirror pairs of certain toroidal orbifolds with discrete torsions can be found in [8], where the mirror symmetry is indeed realized as T-duality on toroidal T^3 fibration in the orbifolds. In these examples the discrete torsions will change under the mirror symmetry transformation.

In [9, 10], Acharya discussed the existence of the mirror symmetry in IIA/IIB string theory compactified on manifolds with exceptional holonomy, including G_2 and $\mathfrak{s}pin(7)$ and argued how the discrete torsion transforms under the T-duality along the four-dimensional torus T^4 fibration. In [1], the authors indeed gave some concrete mirror pairs among Joyce's orbifolds with G_2 holonomy, which are built from resolving or deforming T^7/Z_2^3 orbifolds. They also identified the mirror symmetry as an automorphism in the extended superconformal algebra on manifolds with G_2 holonomy.

Motivated by these known results, we generalize the chiral superconformal algebra to the manifolds with $\mathfrak{s}pin(7)$ holonomy, and identify the corresponding automorphism in the algebra as a combination of a T-duality in 8-direction and a generalized G_2 mirror transformation or a combination of two distinct G_2 mirror transformations. The automorphism could also be understood as T-duality on the supersymmetric T^4 fibrations in the $\mathfrak{s}pin(7)$ manifolds. In order to make the automorphism clearer, we give an example of one class of Joyce's manifolds with $\mathfrak{s}pin(7)$ holonomy and discrete torsions. In these examples there 14 kinds of T-dualities along the T^4 fibration , which can act like the algebra automorphism or mirror symmetry. They are further classified into two categories, one of which does flip the discrete torsion and hence leads to a topologically different Joyce's orbifold while the other does not.

Since the extended superconformal algebra for G_2 and $\mathfrak{s}pin(7)$ manifolds are equivalent to the conformal algebra or the operator product expansion (OPE) of the tri-critical and critical Ising model, we suggest that the mirror symmetry for the exceptional holonomy manifolds might be realized in certain condensed matter system.

We also note that the mirror symmetry in the G_2 manifolds could be realized as a classical operation. Namely it is equivalent to changing the sign of the associative 3-form or reversing the orientation of the associative three cycles. This should be contrasted with the CY mirror symmetry, which is essentially a quantum symmetry involving the worldsheet instanton effects[2].

The paper is organized as follows. In section 2 we will review the mirror symmetry of CY and G_2 manifolds both from the viewpoints of the conformal field theory and the T-duality. In section 3 we will give the construction of $\mathfrak{s}pin(7)$ extended superconformal algebra, identify the automorphism in it as 14 kinds of T-dualities and classify them into two kinds as mentioned above. In section 4 we will give conclusion and discussion, containing some suggestions for the future study.

[2] We would like to thank Edward Witten for pointing this out.

2. Mirror symmetry for CY and G_2 manifolds

To better orientate our readers we give a short review of mirror symmetry on CY orbifolds (T^6/Z_2^2) and Joyce's G_2 manifolds in terms of their sigma model superconformal algebras [1, 3, 4].

2.1. Mirror symmetry of Calabi–Yau threefolds

For CY manifolds, mirror symmetry can be understood, from the viewpoint of the underlying sigma model on CY manifolds, as the effect of a non-trivial automorphism of the sigma model symmetry algebra that is always present for CY compactications, namely, extended superconformal algebra [21]. This approach has also been successfully applied to the manifold with G_2 holonomy [1]. Since the extended superconformal symmetry is so powerful and allows us to gain insights into the structure of the manifolds and construct the space of marginal deformations of the sigma model, we will try to generalize this approach to the manifolds with $\mathfrak{s}pin(7)$ holonomy. Now we start with the review of CY and G_2 cases.

The generators of the $N = 2$ chiral superconformal algebra for string propagation on CY target space are the stress–energy tensor $\mathcal{T}_{\text{CY}}$, two supercurrents $\mathcal{G}_{\text{CY}}, \mathcal{G}'_{\text{CY}}$ of conformal weight $3/2$ and the $U(1)$ current $\mathcal{J}_{\text{CY}}$, along with a complex current Ω_{CY} of conformal weight $3/2$ constructed from the worldsheet fermions (ψ^i and $\tilde{\psi}^i$) and its superpartner Ψ_{CY}. The complex current Ω_{CY} should be thought of as the holomorphic 3-form of the CY. Now we specialize to a class of T^6/Z_2^2 orbifolds, in which the T^6 is the six-dimensional torus, with coordinates $x_i (i = 1, \ldots, 6)$ being periodically identified, $x_i \sim x_i + 1$. The Z_2^2 actions are given by

$$\begin{aligned} \alpha &= (x_1, x_2, -x_3, -x_4, -x_5, -x_6), \\ \beta &= (-x_1, -x_2, x_3, x_4, -x_5, -x_6). \end{aligned} \tag{2.1}$$

The superconformal generators have free field realization and are given by [1, 12, 21]

$$\begin{aligned} &\mathcal{T}_{\text{CY}} = \frac{1}{2}\sum_{j=1}^{6} : \partial x^j \partial x^j : -\frac{1}{2}\sum_{j=1}^{6} : \psi^j \partial \psi^j :, \\ &\mathcal{G}_{\text{CY}} = \sum_{j=1}^{6} : \psi^j \partial x^j :, \quad \mathcal{G}'_{\text{CY}} = \sum_{j=1}^{3} (\psi^{2j-1}\partial x^{2j} - \psi^{2j}\partial x^{2j-1}), \quad \mathcal{J}_{\text{CY}} = \sum_{j=1}^{3} \psi^{2j-1}\psi^{2j}, \\ &\Omega_{\text{CY}} = \psi^1\psi^3\psi^5 - \psi^1\psi^4\psi^6 - \psi^2\psi^3\psi^6 - \psi^2\psi^4\psi^5 \\ &\qquad + i(\psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^5 + \psi^2\psi^3\psi^5 - \psi^2\psi^4\psi^6), \\ &\Psi_{\text{CY}} := \{\mathcal{G}_{\text{CY}}, \Omega_{\text{CY}}\}, \end{aligned} \tag{2.2}$$

where ψ^i are $\tilde{\psi}^i$ are the (1, 1) worldsheet superpartners of x^i for $i = 1, \ldots, 6$. The other half chiral algebra (anti-holomorphic) can be written down similarly in terms of $\bar{\partial}x$ and $\tilde{\psi}$.

There exists a non-trivial automorphism in the superconformal algebra or equivalently the OPE of the operators, which leave the $N = 1$ superconformal subalgebra generated by $\mathcal{T}_{\text{CY}}$ and $\mathcal{G}_{\text{CY}}$ invariant. The automorphism is given by

$$\mathcal{G}'_{\text{CY}} \to -\mathcal{G}'_{\text{CY}}, \quad \mathcal{J} \to -\mathcal{J}, \quad \Omega \to \Omega^*, \quad \Psi \to \Psi^*, \quad \mathcal{T}_{\text{CY}} \to \mathcal{T}_{\text{CY}}, \quad \mathcal{G}_{\text{CY}} \to \mathcal{G}_{\text{CY}}. \tag{2.3}$$

Recall that from the viewpoint of the worldsheet superconformal algebra, the CY mirror symmetry is achieved by applying the aforementioned automorphism (2.3) to one of the chiralities of the algebra, for instance, $\tilde{\mathcal{G}}'_{\mathrm{CY}}$, $\tilde{\mathcal{J}}_{\mathrm{CY}}$, $\tilde{\Omega}_{\mathrm{CY}}$ and $\tilde{\Psi}_{\mathrm{CY}}$. Here the tilde $\sim$ means the anti-holomorphic part of the superconformal algebra. On the other hand, also recall that the T-duality along the ith direction will leave ∂x_i and ψ_i invariant but reverse $\bar{\partial} x_i$ and $\tilde{\psi}_i$. Therefore, we can also easily see that the T-duality on T^3 fibrations in the following directions (which appear in the indices of Ω_{CY}) also generates the mirror symmetry

$$\{(1,3,5),(1,4,6),(2,3,6),(2,4,5),(1,3,6),(1,4,5),(2,3,5),(2,4,6)\}. \tag{2.4}$$

Some concrete examples of these T-dualities acting on the T^3 fibration and changing the discrete torsion can be found in [1, 8].

2.2. *Compact orbifolds with G_2 holonomy*

In this and the following sections, we first give an example of Joyce's orbifolds which were constructed by disingularizing T^7/Z_2^3 and show how the choices in resolving or deforming the singularities can result in topologically different spaces. After that, we will write down the G_2 extended chiral superconformal algebra and look for the automorphisms in it. We will find that applying the automorphism transformation to the superconformal generators with one of two chiralities is equivalent to applying T-dualities along certain T^3 toroidal fibrations. Then these T-dualities will be interpreted as the mirror symmetry transformation.

Consider the orbifolds of T^7/Γ, where T^7 is the seven-dimensional torus with coordinates $x_i (i = 1, \ldots, 7)$ being periodically identified, $x_i \sim x_i + 1$. The discrete Γ is generated by three Z_2 actions given by [3]

$$\begin{aligned}
\alpha &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),\\
\beta &= (-x_1, 1/2 - x_2, x_3, x_4, -x_5, -x_6, -x_7),\\
\gamma &= (-x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7).
\end{aligned} \tag{2.5}$$

In order to desingularize the orbifolds, one has to know, for instance, how the 16 α fixed three-dimensional torus T^3s get identified under the group generated by β and γ. What we found in this example is the 16 T^3s fixed by α or β are reduced to four orbits of order 4 by the free-acting of the $\langle\beta,\gamma\rangle$ or $\langle\gamma,\alpha\rangle$. In the γ-fixed T^3 sector, the group $\langle\alpha,\beta\rangle$ only reduce 16 T^3 to eight orbits of order 2 since $\alpha\beta$ acts trivially on them.

The choices of blowing-up or deforming also come from this γ-fixed sector. From a discrete torsion analysis based on the requirement of modular invariance [1], we know that blowing-up (deforming) corresponds to discrete torsion in the γ-fixed sector $\epsilon_{\gamma;\tilde{f}} = 1\ (-1)$ and the even (odd) $\alpha\beta$ parity. By virtue of the correspondence between the Ramond–Ramond (RR) ground states and the cohomologies, we can write down the RR ground states in the γ-fixed sector. Among 8 γ-fixed T^3s, we can choose $l\alpha\beta$ parity even blowing-ups and $(8-l)\alpha\beta$ parity odd case deformations. We denote such a desingularized manifold by X_l. It has been shown by Joyce that X_l is indeed a manifold with G_2 holonomy [3, 4].

Now we can perform an analysis on how the blowing-up and deformation change the Betti numbers. For $\alpha\beta$ parity even case (blowing-up), we have

$$\epsilon_{\gamma;\tilde{f}} = 1,$$

$$|0,0;\tilde{f}\rangle_\gamma, \qquad \psi^{2+}|0,0;\tilde{f}\rangle_\gamma, \qquad \psi^{4+}\psi^{6+}|0,0;\tilde{f}\rangle_\gamma, \qquad \psi^{2+}\psi^{4+}\psi^{6+}|0,0;\tilde{f}\rangle_\gamma, \tag{2.6}$$

where $\tilde{f} = 1, \ldots, l$ labelling the γ-fixed points after α or β identification.

For $\alpha\beta$ parity odd case (deformation), the RR ground states are

$$\epsilon_{\gamma;\tilde{f}} = -1,$$

$$\psi^{4+}|0,0;\tilde{f}\rangle_\gamma, \qquad \psi^{6+}|0,0;\tilde{f}\rangle_\gamma, \qquad \psi^{2+}\psi^{4+}|0,0;\tilde{f}\rangle_\gamma, \qquad \psi^{2+}\psi^{6+}|0,0;\tilde{f}\rangle_\gamma, \quad (2.7)$$

where $\tilde{f} = l+1,\ldots,8$. One should regard $|0,0;\tilde{f}\rangle_\gamma$ as the harmonic 2-form associated with the exceptional divisors of the blowing-up (deformation). Therefore, blowing-up contributes 1 to b_2 and 1 to b_3 while the deformation increases b_3 by 2.

For the RR ground states in the γ-fixed sector, the operation $\alpha\beta$ reverses the 4th and 6th directions. Therefore, we can express it as

$$\alpha\beta = \frac{1}{4}\psi_0^4\psi_0^6\tilde{\psi}_0^4\tilde{\psi}_0^6\epsilon_{\gamma;\tilde{f}}. \quad (2.8)$$

After summing up the contributions from various sectors, we have the following Betti numbers for X_l:

$$(b_0,\ldots,b_7) = (1,0,8+l,47-l,47-l,8+l,0,1). \quad (2.9)$$

2.3. G_2 extended superconformal algebra

The algebra on manifolds with G_2 holonomy is generated by appending a spin-3/2 operator Φ_{G_2} and its superpartner $\mathcal{X}_{G_2}$ of conformal weight two to the $N=1$ superconformal subalgebra spanned by $\mathcal{T}_{G_2}$ and $\mathcal{G}_{G_2}$ [12, 17]. In our basis of fields x^i and ψ^i, the generators are given by

$$\mathcal{T}_{G_2} = \frac{1}{2}\sum_{j=1}^{7} :\partial x^j\partial x^j: -\frac{1}{2}\sum_{j=1}^{7} :\psi^j\partial\psi^j:, \qquad \mathcal{G}_{G_2} = \sum_{j=1}^{6} :\psi^j\partial x^j:,$$

$$\Phi_{G_2} = \psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^5 + \psi^2\psi^3\psi^5 - \psi^2\psi^4\psi^6 + \psi^1\psi^2\psi^7 + \psi^3\psi^4\psi^7 + \psi^5\psi^6\psi^7,$$

$$\mathcal{X}_{G2} = -\psi^2\psi^4\psi^5\psi^7 - \psi^2\psi^3\psi^6\psi^7 - \psi^1\psi^4\psi^6\psi^7 + \psi^1\psi^3\psi^5\psi^7 - \psi^3\psi^4\psi^5\psi^6$$
$$- \psi^1\psi^2\psi^5\psi^6 - \psi^1\psi^2\psi^3\psi^4 - \frac{1}{2}\sum_{j=1}^{7} :\psi^j\partial\psi^j:. \quad (2.10)$$

However this is not the complete set of the generators of the algebra because we can obtain two new operators $\mathcal{K}_{G_2}$ and $\mathcal{M}_{G_2}$, superpartners of Φ_{G_2} and $\mathcal{X}_{G_2}$, by performing the OPE with $\mathcal{G}_{G_2}$,

$$\begin{aligned} \mathcal{G}_{G_2}(z)\Phi_{G_2}(w) &= \frac{1}{z-w}\mathcal{K}_{G_2}(w) + \cdots \\ \mathcal{G}_{G_2}(z)\mathcal{X}_{G_2}(w) &= -\frac{1}{2}\frac{1}{(z-w)^2}\mathcal{G}_{G_2}(w) + \frac{1}{z-w}\mathcal{M}_{G_2}(w) + \cdots, \end{aligned} \quad (2.11)$$

where the ellipsis refers to the regular parts in the OPE. $\mathcal{K}_{G_2}$ and $\mathcal{M}_{G_2}$ can be explicitly written out in terms of ∂x^i and ψ^i [17] but we omit the expressions since they are tedious and not needed in the following discussion. We also have the antiholomorphic part of the extended superconformal algebra, which is not listed here. The extended superconformal algebra has one obvious but non-trivial automorphism [12, 17]:

$$\Phi_{G2} \to -\Phi_{G_2}; \qquad \mathcal{K}_{G_2} \to -\mathcal{K}_{G_2}; \qquad \mathcal{T}_{G_2},\ \mathcal{G}_{G_2},\ \mathcal{X}_{G_2},\ \mathcal{M}_{G_2} \text{ unchanged}. \quad (2.12)$$

Since geometrically we should think of Φ_{G_2} as the associative 3-form on the G_2 manifold, this mirror symmetry has an interpretation of changing the sign of the 3-form or reversing the orientation of the associative three cycles calibrated by Φ_{G_2}.

If the G_2 manifolds are of the form $(CY_3 \times S^1)/Z_2$ as the Joyce G_2 manifolds, we can also reformulate the superconformal generators in terms of the CY generators as follows:

$$\begin{aligned}
&\mathcal{T}_{G_2} = \mathcal{T}_{\rm CY} + \tfrac{1}{2} : \partial x^7 \partial x^7 : -\tfrac{1}{2} : \psi^7 \partial \psi^7 :, \qquad \mathcal{G}_{G_2} = \mathcal{G}_{\rm CY} + : \psi^7 \partial x^7 :,\\
&\Phi_{G_2} = {\rm Im}(\Omega_{\rm CY}) + : \mathcal{J}_{\rm CY} \psi^7 :,\\
&\mathcal{X}_{G_2} =: {\rm Re}(\Omega_{\rm CY}) \psi^7 : + \tfrac{1}{2} : \mathcal{J}_{\rm CY} \mathcal{J}_{\rm CY} : -\tfrac{1}{2} : \psi^7 \partial \psi^7 :,\\
&\mathcal{K}_{G2} = {\rm Im}(\Psi_{\rm CY}) + : \mathcal{J}_{\rm CY} \partial x^7 : + : \mathcal{G}'_{\rm CY} \psi^7 :,\\
&\mathcal{M}_{G2} =: {\rm Re}(\Psi_{\rm CY}) \psi^7 : - : {\rm Re}(\Omega_{\rm CY}) \partial x^7 : + : \partial x^7 \partial \psi^7 : + : \mathcal{J}_{\rm CY} \mathcal{G}'_{\rm CY} : -\tfrac{1}{2} \partial \mathcal{G}_{\rm CY}.
\end{aligned} \tag{2.13}$$

Similarly, the generalized mirror symmetry for manifolds with G_2 holonomy is to apply the above automorphism to one of the two chiralities. On the other hand, the T-duality in the following (i_1, i_2, i_3) directions can obviously realize the automorphism:

$$\begin{aligned}
&(i_1, i_2, i_3) \in I_3^+ \cup I_3^-,\\
&I_3^+ = \{(2, 4, 6), (2, 3, 5), (1, 2, 7)\},\\
&I_3^- = \{(1, 3, 6), (1, 4, 5), (3, 4, 7), (5, 6, 7)\}.
\end{aligned} \tag{2.14}$$

If we combine any two different T-dualities in (2.14), we obtain another set of T-dualities acting on toroidal T^4, which also leave the extended chiral algebra invariant. Hence, they are mirror symmetries which will take IIA (IIB) to IIA (IIB):

$$\begin{aligned}
&(i_1, i_2, i_3, i_4) \in I_4^+ \cup I_4^-,\\
&I_4^+ = \{(1, 3, 5, 7), (1, 4, 6, 7), (3, 4, 5, 6)\},\\
&I_4^- = \{(2, 4, 5, 7), (2, 3, 6, 7), (1, 2, 5, 6), (1, 2, 3, 4)\}.
\end{aligned} \tag{2.15}$$

Recall that T-duality in the ith direction will give $\tilde{\psi}^i$ a minus sign. It is not hard to see that I_3^+ (I_4^+) does not change the discrete torsion while I_3^- (I_4^-) does. We summarize mirror symmetry on G_2 holonomy manifold according to the actions of the T-dualities along $I_3^{\pm}$ as follows:

$$\begin{aligned}
&IIA(IIB)/X_l \longleftrightarrow IIB(IIA)/X_{8-l}, \ \text{under } I_3^-,\\
&IIA(IIB)/X_l \longleftrightarrow IIA(IIB)/X_l, \ \text{under } I_3^+.
\end{aligned} \tag{2.16}$$

3. Mirror symmetry for $\mathfrak{spin}(7)$ manifolds

3.1. Joyce's construction of $\mathfrak{spin}(7)$ manifolds

There are many known examples of Joyce's $\mathfrak{spin}(7)$ orbifolds [2]. For simplicity, let us take one class of Joyce's orbifolds in which we have multiple choices in desingularizing T^8/Z_2^4 as above, so that the T-duality will exchange different desingularizations. Now let us take T^8/Z_2^4, where T^8 is the eight-dimensional torus with all periodicities 1. And the Z_2^4 generators act as follows:

$$\begin{aligned}
&\alpha = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8),\\
&\beta = (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8),\\
&\gamma = (1/2 - x_1, -x_2, x_3, x_4, 1/2 - x_5, -x_6, x_7, x_8),\\
&\delta = (-x_1, x_2, 1/2 - x_3, x_4, 1/2 - x_5, x_6, 1/2 - x_7, x_8).
\end{aligned} \tag{3.1}$$

Again, the periodicity of x_i is set to be unity. In general, the singularities arises in five different types and the corresponding desingularizations are listed as follows.

Type (1): increase b_2 by 1, b_3 by 4, b_{4+} by 3 and b_{4-} by 3. The singularity type is $T^4 \times \left(B^4_\epsilon/\{\pm 1\}\right)$, where B^4_ϵ is defined as an open ball of radius ϵ about 0 in R^4.

Type (2): increase b_2 by 1, b_{4+} by 3 and b_{4-} by 3. The singularity if of the form $\left(T^4/\{\pm 1\} \times \left(B^4_\epsilon/\{\pm 1\}\right)\right.$.

Type (3): increase b_{4+} by 1. The singularity is $\left(B^4_\epsilon/\{\pm 1\} \times \left(B^4_\epsilon/\{\pm 1\}\right)\right.$.

Type (4A): increase b_2 by 1, b_3 by 2, b_{4+} by 1 and b_{4-} by 1.

Type (4B): increase b_3 by 2, b_{4+} by 2 and b_{4-} by 2. The singularity of type(4) is an isometric involution σ of $T^4 \times \left(B^4_\epsilon/\{\pm 1\}\right)$, where $\sigma = (1/2 + x_1, x_2, -x_3, -x_4, y_1, y_2, -y_3, -y_4)$. Namely, the singular set is isomorphic to $\left(T^4 \times \left(B^4_\epsilon/\{\pm 1\}\right)\right)/\langle \sigma\rangle$.

Type (5A): increase b_2 by 1, b_{4+} by 1 and b_{4-} by 1.

Type (5B): increase b_{4+} by 2, and b_{4-} by 2. The singularity of type(5) is isomorphic to $\left(T^4/\{\pm 1\} \times B^4_\epsilon/\{\pm 1\}\right)/\langle \sigma\rangle$.

As a result, one finds the singular set of this orbifold contains 2 type(1), 8 type(2), 64 type(3) and 4 type(4). If we choose to have j type(4A) and $(4-j)$ type(4B) and add up all the Betti numbers in the twisted sectors as well as the untwisted sector, we have Joyce's manifolds Y_j with

$$b_2 = 10 + j, \quad b_3 = 16, \quad b_{4+} = 109 - j, \quad b_{4-} = 45 - j, \quad j = 0, \ldots, 4 \tag{3.2}$$

$$\hat{A} = \tfrac{1}{24}(-1 + b_1 - b_2 + b_3 + b_{4+} - 2b_{4-}) = 1. \tag{3.3}$$

It was shown by Joyce that Y_j is compact manifolds with $\mathfrak{s}pin(7)$ holonomy [2], and therefore, they are suitable for string theory compactification. To be more precise, the 4 type(4) singularities come from 16 γ-fixed 4 torus T^4s. Note that $\alpha\delta$ acts trivially on these T^4s and the group elements $\alpha, \beta, \alpha\beta$ and $\beta\delta$ act freely on them and reduce the number of T^4s to 4. Therefore, we have RR ground states $|0, 0; \tilde{f} = 1, 2, 3, 4\rangle_\gamma$ corresponding to the harmonic two forms of the exceptional divisors. Similarly, the $\alpha\delta$ parity of $|0, 0; \tilde{f}\rangle_\gamma$ is also given by the discrete torsion $\epsilon_{\gamma,\tilde{f}}$. Since the action of $\alpha\delta$ inverses directions 4 and 7, we can construct RR ground states accordingly as follows.

For $\alpha\delta$ parity even case, we have

$$\begin{gathered}\epsilon_{\gamma;\tilde{f}} = 1,\\ |0,0;\tilde{f}\rangle_\gamma, \quad \psi^{3+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{8+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{3+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{4+}\psi^{7+}|0,0;\tilde{f}\rangle_\gamma,\\ \psi^{3+}\psi^{4+}\psi^{7+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{4+}\psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{3+}\psi^{4+}\psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma.\end{gathered} \tag{3.4}$$

For the $\alpha\delta$ parity odd case, the RR ground states are

$$\begin{gathered}\epsilon_{\gamma;\tilde{f}} = -1,\\ \psi^{4+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{7+}|0,0;\tilde{f}\rangle_\gamma,\\ \psi^{3+}\psi^{4+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{3+}\psi^{7+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{4+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma,\\ \psi^{3+}\psi^{4+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma, \quad \psi^{3+}\psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma.\end{gathered} \tag{3.5}$$

Obviously, we obtain $\Delta b_2 = 1, \Delta b_3 = 2, \Delta b_4 = 2$ in the parity even case, and $\Delta b_3 = 2, \Delta b_4 = 4$ in the parity odd case, which agrees with the mathematical analysis in [2]. Taking j type(4A) and $(4-j)$ type(4B) and collecting all the Betti number contributions from various places, we obtain the Betti numbers in (3.2).

3.2. $\mathfrak{spin}(7)$ extended superconformal algebra

Now consider a direct product space $M \times S^1$, where M is a manifold with G_2 holonomy. It is always possible to define a $\mathfrak{spin}(7)$ structure. And the Cayley 4-form ϕ_4 in this manifold with $\mathfrak{spin}(7)$ structure can be written as

$$\phi_4 = *\phi_3 + \phi_3 \wedge \mathrm{d}x^8, \tag{3.6}$$

where ϕ_3 is the calibrated 3-form in the G_2 manifold.

It is also true that in this example in the previous section $T^7/\langle\alpha, \gamma, \delta\rangle$ gives rise to Joyce's 7-manifold of G_2 holonomy, if we forget about 8-direction. In fact, we can have a more generic statement that $T^7/\langle\alpha, \gamma, \delta\rangle$ is *always* a manifold with G_2 holonomy for any half-integer choices of the constants c_i [11], where

$$\begin{aligned}
\alpha &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7),\\
\gamma &= (c_1 - x_1, c_2 - x_2, x_3, x_4, c_5 - x_5, c_6 - x_6, x_7),\\
\delta &= (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, c_7 - x_7).
\end{aligned} \tag{3.7}$$

If we reformulate the action of β in the previous section, we will find that β acts as

$$\beta : x^8 \to -x^8, \qquad \beta^*(\phi_3) = -\phi_3, \qquad \beta^*(*\phi_3) = *\phi_3. \tag{3.8}$$

In this example, Z_2 action β turns the $\mathfrak{spin}(7)$ structure into the $\mathfrak{spin}(7)$ holonomy. However, it is not clear that we can always form manifolds with $\mathfrak{spin}(7)$ holonomy by modding out this kind of Z_2 involution on $G_2 \times S^1$.

Therefore, at least in Joyce's orbifolds, relation (3.6) enables us to write down the expression of the stress–energy tensor $\mathcal{T}_{\mathrm{spin}(7)}$, the supercurrent $\mathcal{G}_{\mathrm{spin}(7)}$, a spin-2 operator $\mathcal{X}_{\mathrm{spin}(7)}$ and its spin-5/2 superpartner $\mathcal{M}_{\mathrm{spin}(7)}$ in terms of the corresponding quantities in G_2 manifolds [17]:

$$\begin{aligned}
\mathcal{T}_{\mathrm{spin}(7)} &= \mathcal{T}_{G_2} + \tfrac{1}{2} : \partial x^8 \partial x^8 : - \tfrac{1}{2} : \psi^8 \partial \psi^8 :,\\
\mathcal{G}_{\mathrm{spin}(7)} &= \mathcal{G}_{G_2} + : \psi^8 \partial x^8 :,\\
\mathcal{X}_{\mathrm{spin}(7)} &= \mathcal{X}_{G_2} + \Phi_{G_2} \psi^8 + \tfrac{1}{2} \psi^8 \partial \psi^8,\\
\mathcal{M}_{\mathrm{spin}(7)} &= [\mathcal{G}_{\mathrm{spin}(7)}, \mathcal{X}_{\mathrm{spin}(7)}]\\
&= \partial x^8 \Phi_{G_2} - \mathcal{K}_{G_2} \psi^8 - \mathcal{M}_{G_2} + \tfrac{1}{2} \partial^2 x^8 \psi^8 - \tfrac{1}{2} \partial x^8 \partial \psi^8.
\end{aligned} \tag{3.9}$$

From these generators for the extended supercomformal algebra, it is not difficult to see that the combination of the G_2 automorphism (2.12) and the T-duality in 8-direction is an automorphism in the superconformal algebra. In addition, the T-duality in (2.15) is also an automorphism in the algebra. Therefore, we have a list of 14 T-dualities on T^4 toroidal fibrations which generate the mirror symmetry

$$\begin{aligned}
\{&(2, 4, 6, 8), (2, 3, 5, 8), (1, 2, 7, 8), (1, 3, 6, 8), (1, 4, 5, 8), (3, 4, 7, 8),\\
&(5, 6, 7, 8), (1, 3, 5, 7), (1, 4, 6, 7), (3, 4, 5, 6), (2, 4, 5, 7), (2, 3, 6, 7),\\
&(1, 2, 5, 6), (1, 2, 3, 4)\}.
\end{aligned} \tag{3.10}$$

The first line consists of T-dualities in directions in (2.14) and 8-direction. The second line is the same as the directions listed in (2.15). In this $\mathfrak{spin}(7)$ case, we do not have the similar relation like (2.12). Therefore, in order to visualize the automorphism in the superconformal algebra, we have to express the $\mathfrak{spin}(7)$ generators and the algebra in terms of G_2 generators and construct our desirable mirror transformation from G_2 automorphism (2.12), (2.14) and (2.15). Finally, the expression of $\alpha\delta$ in the γ-fixed sector is

$$\alpha\delta = \tfrac{1}{4}\psi_0^4\psi_0^7\tilde{\psi}_0^4\tilde{\psi}_0^7\epsilon_{\gamma;\tilde{f}}. \tag{3.11}$$

By the similar reasoning, the 14 T-dualities are divided into two sets $J_4^{\pm}$:

$$\begin{aligned}&(i_1, i_2, i_3, i_4) \in J_4^+ \cup J_4^-,\\ &J_4^+ = \{(2,3,5,8), (1,3,6,8), (3,4,7,8), (1,4,6,7), (2,4,5,7), (1,2,5,6)\},\\ &J_4^- = \{(2,4,6,8), (1,2,7,8), (1,4,5,8), (5,6,7,8), (1,3,5,7), (3,4,5,6),\\ &\qquad (2,3,6,7), (1,2,3,4)\}.\end{aligned} \tag{3.12}$$

The T-duality actions in $J_4^{\pm}$ on type-II superstring theory are thus summarized by

$$\begin{aligned}&IIA(IIB)/Y_j \longleftrightarrow IIA(IIB)/Y_{4-j}, \text{ under } J_4^-,\\ &IIA(IIB)/Y_j \longleftrightarrow IIA(IIB)/Y_j, \text{ under } J_4^+.\end{aligned} \tag{3.13}$$

So far we only consider one class of the $\mathfrak{spin}(7)$ manifolds constructed by Joyce, but we believe that there should exist many more other examples of the mirror $\mathfrak{spin}(7)$ pairs.

4. Conclusion and discussion

In this paper we have generalized the construction of [1] to Joyce's manifolds with $\mathfrak{spin}(7)$ holonomy and shown, in a class of the examples of Joyce's $\mathfrak{spin}(7)$ manifolds, how the mirror symmetry is realized in the worldsheet superconformal algebra as a combination of a T-duality in 8-direction and a G_2 mirror symmetry transformation, or a combination of two distinct G_2 mirror transformations. The $\mathfrak{spin}(7)$ mirror transformation contains 14 different kinds of T-dualities on the supersymmetric T^4 fibrations. By an analysis on the changes of discrete torsions in the RR sector, we classify these 14 T-dualities into 2 sorts, one of which changes the discrete torsions and the other does not. The T-dualities which will flip the discrete torsions will then change the Betti number of the manifolds.

Another interesting fact is that the extended worldsheet superconformal algebra on G_2 manifolds is equivalent to the OPE of the tri-critical Ising model with central charge 7/10, while the $\mathfrak{spin}(7)$ superconformal algebra is the same as the Ising model OPE [17]. In both cases, the higher dimensional operators play the role of higher spin operators in the W-algebras. It is natural to propose that the mirror symmetries discussed in this paper can be realized as certain duality phenomena in the condensed matter system! It will be very interesting to understand this better.

We also note that the mirror symmetry in the G_2 manifolds could be realized as a classical operation. Namely it is equivalent to reversing the orientation of the associative three cycles. This should be contrasted with the CY mirror symmetry, which is essentially a quantum symmetry. It is very likely that the $\mathfrak{spin}(7)$ mirror symmetry is also a classical operation. However, it is still subject to further investigation.

In [13], the authors completed a cycle of the dualities by explicitly performing the T-duality on T^3 fibration and a G_2 flop in M-theory. It would be interesting to generalize the computation to a duality cycle involving $\mathfrak{spin}(7)$ and G_2 manifolds and understand how the generalized mirror symmetry lies in this picture [15].

In order to understand the $G_2/\mathfrak{s}pin(7)$ mirror symmetry better, one could try to T-dualize the known various non-compact metric solutions with $G_2/\mathfrak{s}pin(7)$ holonomy [16] and see how they are connected through mirror symmetry. In the CY case, NS-NS fluxes can turn the CY target space into half-flat [14] . The generalized mirror symmetry for G_2 and $\mathfrak{s}pin(7)$ in the presence of the background fluxes also demands some further study. Finally, it would also be interesting to see how we can fit the G_2 or $\mathfrak{s}pin(7)$ mirror symmetry into the correspondence of heterotic(G_2)/ IIA(G_2 orientifold)/M-theory($\mathfrak{s}pin(7)$) [5] . It would be interesting to consider the topological twisted worldsheet sigma model on $\mathfrak{s}pin(7)$ manifolds. Some related construction on the G_2 manifolds has been provided in [22]. The worldsheet topological model study will definitely unravel more of the unknown features of the generalized mirror symmetry in this paper!

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